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(Residential Autonomous College affiliated to University of Calcutta)

M.A./M.Sc. THIRD SEMESTER EXAMINATION, DECEMBER 2015

SECOND YEAR [BATCH 2014-16]

MATHEMATICS

Date : 18/12/2015

Time : 11 am – 1 pm

Paper : XIII [Number Theory]

Full Marks : 50

[The paper carries **58 marks** and you can score maximum **50**. You state the result or theorem clearly whichever you use.]

1. Show that the equation $7x^3 + 2 = y^3$ has no solution in integers. [4]

2. State Chinese Remainder Theorem. Use it to prove the following :

   Let f(x) be a polynomial with integral coefficients, and for positive integer m let N(m) denote the number of incongruent solutions of the congruence $f(x) \equiv 0 \pmod{m}$. Prove that $N(\cdot)$ is a multiplicative function i.e. Prove that $N(m_1m_2) = N(m_1)N(m_2)$ whenever $(m_1, m_2) = 1$. [2+6]

3. Find the number of zeros with which the decimal representation of 100! terminates. Justify your answer. [4]

4. State Hensel's lemma. Use it to solve the following congruence equation :

   $x^2 + x + 47 \equiv 0 \pmod{7^3}$. [2+5]

   Or,

   State quadratic Reciprocity law for Legendre symbols. Use it to examine whether 29 is a quadratic residue module 53. [2+5]

5. If x is real, $x \geq 1$, let $\phi(x, n)$ denote the number of positive integers $\leq x$ that are relatively prime to n.

   Prove that $\phi(x, n) = \sum_{d|n} \mu(d)\left[\frac{x}{d}\right]$ and $\sum_{d|n} \phi\left(\frac{x}{d}, \frac{n}{d}\right) = [x]$. [4+3]

6. If $x \geq 2$, prove that $\sum_{n \leq x} \frac{d(n)}{n} = \frac{1}{2}(\log x)^2 + 2\gamma \log x + O(1)$, where γ is Euler's constant & d is the divisor function. [7]

7. Define Dirichlet character modulo k (a positive integer). List all the Dirichlet characters modulo 6 and 7. [2+3+3]

8. If (h, k) = 1, $h, k \in \mathbb{N}$, prove that there is a constant A (depending on h & k) such that, if $x \geq 2$,

   $$\sum_{\substack{p \leq x \\ p \equiv h(k)}} \frac{1}{p} = \frac{1}{\phi(k)} \log\log x + A + O\left(\frac{1}{\log x}\right).$$

   You may assume the asymptotic formula $\sum_{p \leq x} \frac{1}{p} = \log\log x + O(1)$. [8]

9. If p is a prime and $p \equiv 1 \pmod{4}$ then show that $\sum_{a=1}^{p-1} a\left(\frac{a}{p}\right) = 0$. [5]

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